STUDENT RESPONSE TO ANNUITY FORMULA DERIVATION

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ABSTRACT

This paper presents an intuitive approach to deriving annuity formulas for classroom use and attempts to assess student response to such presentation in a principles of finance course by observing student behavior and surveying student sentiment. After presenting three derivations of an annuity formula, a quiz was administered to determine the correct use of a particular derivation of the student's choosing. The derivation used and its correct use by students is reported. In addition, a survey is conducted to assess student sentiment. Implications for instructors are also discussed.

INTRODUCTION AND LITERATURE REVIEW

This paper attempts to assess student reaction to teaching the derivation of annuity formulas in a principles of finance course. The research methods include observing student behavior and surveying student sentiment. The results presented in this paper allow the instructor some insight into student perceptions of how annuity formulas should be taught.

Many students find the use of mathematics in any discipline somewhat difficult. This is indeed true for finance courses as well, especially for the finance principles course that is usually required of all business majors. Some instructors allow the use of formula sheets during testing. For those students that are not given such luxury, memorizing and comprehending mathematical formulas can be a daunting task. Many students distinguish between material that is more "concept" and that which requires the use of formulas. They fail to realize that the two are inseparable; formulas are simply concepts written in mathematical terms. If students were allowed to see the reasoning behind the formulas, they may better be able to understand and use those formulas.

Time Value of Money (TVM), the valuing of cash flows through time, is a major topic in the teaching of finance. Of the various formulas available to adjust dollar values, those involving the present and future value of annuities may be regarded as perhaps the most difficult to comprehend in the principles course. This may be due to the fact that they are lengthy and their logical derivation is not so easily grasped in the way they are usually presented. Most corporate finance texts write the Present Value of an Annuity (PVA) formula using a constant rate for each payment period in the form

$$PVA = PMT \left[\frac{1 - \frac{1}{(1+r)^n}}{r} \right]$$
(1)

where PVA is the value of the annuity one period before the first payment, PMT is the annuity payment at the end of each period, r is the rate of change in dollar values per period, and n is the number of periods. Some other finance texts use other forms of the formula, some of which will be developed shortly. A common form of the Future Value of an Annuity formula is

$$FVA = PMT\left[\frac{(1+r)^n - 1}{r}\right]$$
(2)

To a student with a limited math background, these formulas can look overwhelming in their concept and size. In an age without computers Sherritt (1944) suggests limiting the number of formulas, restricting the number of symbols across all formulas, and not requiring that formulas be developed. To further assist student learning he graphically connects each formula to a time line diagram.

In order to avoid referring to a geometric progression to explain annuity formulas Watson (1936) instead points out that the FVA formula is the compound interest of \$1 divided by the interest rate per period and that the PVA formula is the compound discount on \$1 divided by the interest rate per period.

In order to help students match a set of cash flows with an appropriate valuation formula Skinner (1994) begins with a geometric progression and derives a general "source" formula that can, by imposing different assumptions related to a set of cash flows, be used to calculate the present value of

an ordinary annuity a perpetuity a growing perpetuity a growing annuity a growing annuity with constant rates of growth.

Using this formula with varying sets of assumptions, "establishes the critical link between the structure of the cash flow to be valued and the appropriate model to be used" (Skinner, 1994, p. 87).

Most introductory finance texts simply omit any explanation of the annuity formulas. A few, however, do present to varying degrees the concept that an annuity is a piece of a perpetuity, and then use that concept to derive the annuity formulas. Notable texts that offer such presentations either in the main text, footnotes, or as web extensions or chapters include Brealey, Myers, and Allen (2014), Brigham and Ehrhardt (2014), Ross, Westerfield, and Jaffe (2013), and Welch (2009). Although seemingly overwhelming to a student in introductory finance the formulas actually represent this quite simple concept of valuing an annuity as a part of a perpetuity.

The connection an annuity has with a perpetuity is not easily seen in the often used forms of annuity formulas. However, assuming that students easily accept the perpetuity formula they may be better prepared to accept annuity formulas if they see that such formulas can derived from the simple perpetuity formula. Newcombe et al. (2009) cite several studies in which elementary school children were better able to determine solutions to fraction problems when they were able to connect such problems to "related informal experiences." Newcombe et al. also cite research that shows that confidence in one's math ability increases performance in science, technology, engineering, and mathematics (STEM) learning activities. Beginning with the simple perpetuity formula may provide students with familiar material that can be used to build confidence to use the more difficult annuity formulas.

The major question this paper asks is, "How do students respond to a presentation of the derivation of an annuity formula from the perpetuity formula?" This paper uses the method of observing student behavior and surveying student sentiment. First, students are presented with the derivation of the annuity formulas that is similar to Brealey, Myers, and Allen (2014), Brigham and Ehrhardt (2014), Ross, Westerfield, and Jaffe (2013), and Welch (2009). Three common forms of each of the formulas are given, with each form representing regressive "closeness" to the simple concept from which it was derived. As will be shown shortly, the first form is written in a way that better reflects the concept of its derivation (the most intuitive form). The subsequent two forms, each a derivation of the previous form, progressively distance themselves from the initial simple derivation (less intuitive forms). After presenting the three forms, students are given a TVM quiz which requires them to write the formulas they used.

However, although some students may use one form, they still may have benefited from being exposed to other forms. This cannot be detected by observation. Therefore, students are asked if they did receive some benefit from being taught how the annuity formulas were derived. Both negative and positive reactions are anticipated. Although some students may benefit, others may deem such derivations too complex, deep, and confusing. By revealing the benefits and costs involved through student usage and perceptions, the results presented in this paper allow the TVM instructor some insight into how the annuity formulas should be taught. Such analysis of students' response to formula derivation is lacking in the literature.

DERIVING THE ANNUITY FORMULAS

Before the results are reported, the derivation of the PVA formula into three forms is given as a sample of what students were presented. One of the simplest formulas in finance is that of the Present Value of a Perpetuity (PVP). The PVP formula is

$$PVP = \frac{PMT}{r}$$
(3)

where PMT is the perpetual periodic cash flow, and r is the discount percentage rate. The presentation presumes students understand this simple formula.

The simple concept in deriving the present value of an annuity results from the fact that the annuity is simply a part of a perpetuity; the only difference between a perpetuity and an annuity is that a perpetuity has an infinite stream of payments whereas an annuity has a finite stream of payments. The object of deriving the annuity formula from the perpetuity formula rests in

subtracting the part of the perpetuity that is not needed in order to leave the part of the perpetuity that is needed, the annuity. This is illustrated in Figure 1. Perpetuity B is subtracted from Perpetuity A to result in Annuity Z. Since the part of Perpetuity A that is not needed extends into infinity the subtracted part is, itself, is an annuity (Perpetuity B). The remaining payments of Perpetuity A (that no longer extend into infinity) would form the desired annuity, Annuity Z. Of course, in order to subtract the value of the future perpetuity from the value of the present perpetuity, the two values must be at the same time period. The two values must be adjusted to the desired time period, and the resulting difference would be the value of the annuity at that time period.

This process is illustrated in more detail in Figure 2 by deriving the present value of a fourperiod annuity (value at t = 0; n = 4). The value of a perpetuity at period 0 is calculated. The value of the fifth and remaining payments needs to be subtracted in order to retain and reveal the value of the four-period annuity. This is done by first calculating the value at period 4 (t + n) of those unwanted payments. Since the unwanted payments also form a perpetuity, this results in the value of that perpetuity at period 4. Since the value of the annuity is to be calculated at period 0, the value of the unwanted payments, PVP4, is adjusted to period 0 using the simple present value formula (which would have been previously presented in class). The value at period 0 of the desired annuity can then be found by subtracting PVP4 from PVP0. Thus the Present Value of an Annuity (PVA) can be calculated as

$$PVA_0 = \frac{PMT}{r} - \frac{PMT}{r} \left[\frac{1}{(1+r)^4} \right]$$
(4)

Or more generally

$$PVA_{t} = \frac{PMT}{r} - \frac{PMT}{r} \left[\frac{1}{(1+r)^{n}} \right]$$
(5a)

where n is the number of payment periods proceeding (to the future of) time period t. This is the first form of the PVP formula and will be designated as Form A. The concept of how the





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annuity formula is derived is best seen in this form. Rearranging Form A results in forms B and C, respectively,

FORM B:
$$PVA_t = \frac{PMT}{r} \left[1 - \left(\frac{1}{(1+r)^n} \right) \right]$$
 (5b)

FORM C: PVA_t

$$= PMT \left[\frac{1 - \frac{1}{(1+r)^n}}{r} \right]$$
(5c)

Notice that Equation 5c (Form C) is the same as Equation 1 presented earlier. A similar presentation is given to derive three forms of the Future Value of an Annuity (FVA) which are not presented here.

METHODOLOGY

Students were first presented the derivations of the future and present value annuity formulas. Students were not required to derive the annuity formulas at any time during the semester. Within a week of when the annuity formula derivations were taught, students were given a pop quiz to test their ability to calculate: (1) present value of an annuity (PVA), (2) future value of an annuity (FVA), (3) the payment of an annuity given its present value (PMT), (4) present value of a perpetuity (PVP). Having the students calculate the payment for an annuity is included to assess student reaction of having to manipulate any given form to solve for a missing variable. Although students could use a financial calculator, they were required to write the formula used in solving the problems.

Hypothesis tests are used to determine if any significant difference exists between the following:

- 1. The proportion of students who recalled a given formula and those who did not.
- 2. Of those who could recall a given formula, the proportion of students who recalled a particular form of that formula and those who recalled another form.
- 3. Of those who could recall a given formula, the proportion of students who calculated the correct answer and those who could not.
- 4. Of those who could recall a given form of a formula, the proportion of students who calculated the correct answer and recalled a particular form of a formula and those who calculated the correct answer and recalled another form.

Although students may use one of the three forms, they still may have benefited by being exposed to the two other forms. For instance, a student may use form C, but the student may be better able to use that form in calculating the correct answer because they were exposed to forms A and B and to their derivations. Therefore, a survey of student sentiment was conducted. Within the last two weeks of the semester, students were asked to (1) write the PVA and FVA formulas

in the form of their choosing together with the PVP formula, and (2) state in writing if exposure to the derivation of the annuity formulas benefited them in their classroom performance. Chi-square and related hypothesis tests are used to determine significance in regard to recalling a particular formula, a particular form of a formula, and if the student responded with a positive, negative, or neutral sentiment to the annuity formula derivations.

RESULTS

Table 1 presents the results of the pop quiz. A total of 86 students took the quiz. Of those students, a total of 77 (90%) were able to correctly write the PVP formula, more than the number who could correctly write the PVA or FVA formulas. This may be a somewhat obvious indication that the PVP formula is much easier to comprehend than the PVA or FVA formulas. Part B of Table 1 shows that of those who remembered the formulas, 90% actually correctly calculated the answer to the FVA problem compared to only 85% for the PVA problem. Only 71% who remembered the PVA formula could use it to solve for the annuity payment. Also, 95% of those who remembered the PVP problem.

The heart of this study is to assess student perceptions of the derivations of the annuity formulas. Part A of Table 1 reports the number (and percentage) of students who correctly remembered each of the three forms. Part B of Table 1, reflecting the information presented in Part A, reports the number (and percentage) of students answering the problems correctly using each of the three forms. The majority of students favored Form C, followed by Form A and then Form B in all three types of annuity problems. The proportion of students who used Form B in computing the PVA (15%) is significantly less than those who used either Form A (38%) or Form C (48%). No significant difference is found between the proportion of students who used Form A and those who used Form C. However, when used to compute the FVA or the payment for the PVA, Form C is used significantly more than either of the other two forms.

Table 1C shows that while students who used Form A were the least able to calculate the correct answers for the PVA problem, no significant difference exists in the correct use of each form. Although a significant proportion of students used Form C in the FVA problem, the proportion of students correctly using either Form A or Form B was significantly greater than those using Form C. In other words, more students used Form C, but a larger proportion of students used Forms A and B correctly. The form that provided the best performance in calculating the payment for the PVA problem was Form C, followed closely by Form B. This seems reasonable, since using Form C or Form B involves less manipulation of the PVA formula to isolate the annuity payment (PMT).

Table 1							
		Quiz Results		DVD			
	PVA	FVA	PM1 for PVA	PVP			
Total Attempts	86	86	86	86			
Total correct formulas / total	(1)()()()()()()()()()()()()()()()()()()		50/06/600/04/4				
attempts (%) ^a :	61/86 (71%)***	62/86 (72%)***	59/86 (69%)***	77//86 (90%)***			
A2.	PVA	FVA	PMT for PVA	PVP			
Total correct forms / total							
correct formulas (%):							
Form A	23/61 (38%)***	16/62 (26%)***	17/59 (29%)***	-			
Form B	9/61 (15%)***	10/62 (16%)***	10/59 (17%)***	-			
Form C	29/61 (48%)***	36/62 (58%)***	32/59 (54%)***	-			
Pair-wise Comparisons ^b :							
Form A vs. Form B	38%>15%***	26% > 16%***	29% > 17%***	-			
Form A vs. Form C	38% < 48%***	26% < 58% ***	29% < 54%***	-			
Form B vs. Form C	15% < 48%***	16% < 58%***	17% < 54%***	-			
B1.	PVA	FVA	PMT for PVA	PVP			
Total correct problems / total							
formulas (%) ^a :	52/61 (85%)***	56/62 (90%)***	42/59 (71%)***	73/77 (95%)***			
B2.	PVA	FVA	PMT for PVA	PVP			
Total correct problems / total							
correct form (%):							
Form A	18/23 (78%)***	16/16 (100%)**	7/17 (41%)***	-			
Form B	8/9 (89%)***	10/10 (100%)**	8/10 (80%)***	-			
Form C	26/29 (90%)***	30/36 (83%)**	27/32 (84%)***	-			
Pair-wise Comparisons ^b :							
Form A vs. Form B	78% < 89%***	100% ≠ 100%***	41% < 80%***	-			
Form A vs. Form C	78% < 90%***	100%>83%***	41% < 84%***	-			
Form B vs. Form C	89% < 90%***	100% > 83%***	80% < 84%***	-			
***, **, * indicate significance at the 1, 5, and 10 percent, respectively							
^a Chi-Square test is used							
^b Marascuillo procedure is used	^b Marascuillo procedure is used						

Table 2 reports students' recall of the formulas and their perceptions of how the formulas were presented in the classroom. Students were asked if they benefited from the annuity formula derivation lecture and were also asked the write down the formulas for PVA, FVA, and PVP. Seventy-seven students were surveyed in total. A large number of students (24 out of 77, or 31%) did not state their perceptions while others gave either an unclear or neutral response. The vast majority had a favorable impression of the classroom presentation; 47 students (61%) stated that they were positively affected from having the derivation of the formulas presented to them in class. Only 6 students (8%) stated that they were negatively affected.

Table 2 Survey Results						
A .	Positive	Negative	N/A*	Total		
Total Response	47/77 (61%)	6/77 (8%)	24/77 (31%)	77		
Pair-wise Comparisons ^b :						
Positive vs Negative	61% > 8%***					
Positive vs N/A	61% > 31% ***					
Negative vs N/A	8% < 31%***					
B1.	Positive	Negative	N/A*	Total		
PVA: Total Formulas Recalled	30/47 (64%)	1/6 (17%)	15/24 (63%)	46/77(60%)		
Pair-wise Comparisons ^b		1,0 (1,70)	10/2 (00/0)	10,77(0070)		
Positive vs Negative	64% > 17%***					
Positive vs N/A	64% > 63% ***					
Negative vs. N/A	17% < 63%***					
B2	Positive	Negative	N/A*	Total		
PVA: (Total correct formulas /	1 0010110		1011	1000		
total response)						
Form A	8/30 (27%)	0/1 (0%)	2/15 (13%)	10/46 (22%)		
Form B	6/30 (20%)	0/1 (0%)	0/15 (0%)	6/46 (13%)		
Form C	16/30 (53%)	1/1 (100%)	13/15 (87%)	30/46 (65%)		
Pair-wise Comparisons ^{b.}	10/00 (00/0)	1/1 (10070)	15/15 (0770)			
Form A vs. Form B	27% > 20%***	$0\% \neq 0\%^{***}$	13% > 0%***	22% > 13%***		
Form A vs. Form C	27% < 53%***	0% < 100%***	13% < 87%***	22% < 65%***		
Form B vs. Form C	$20\% < 53\%^{***}$	0% < 100%***	0% < 87%***	13% < 65%***		
C1.	Positive	Negative	N/A*	Total		
EVA: Total Formulas Recalled	26/47 (55%)	1/6 (17%)	11/24 (46%)	38/77 (49%)		
Pair-wise Comparisons ^b :	20/17 (00/0)	1/0 (1//0)	11/21 (10/0)			
Positive vs. Negative	55% > 17%***					
Positive vs. N/A	55% > 46%***					
Negative vs. N/A	17% < 46%***					
C2.	Positive	Negative	N/A*	Total		
FVA: (Total correct formulas /						
total response)						
Form A	8/26 (31%)	0/1 (0%)	1/11 (9%)	9/39 (23%)		
Form B	6/26 (23%)	0/1 (0%)	1/11 (9%)	7/39 (18%)		
Form C	12/26 (46%)	1/1 (100%)	9/11 (82%)	23/39 (59%)		
Pair-wise Comparisons ^b :				. ,		
Form A vs. Form B	31% > 23%***	0% ≠ 0%***	9% ≠ 9%***	23% > 18%***		
Form A vs. Form C	31% < 46%***	0% < 100%***	9% < 82%***	23% < 59%***		
Form B vs. Form C	23% < 46%***	0% < 100%***	9% < 82%***	18% < 59%***		
D.	Positive	Negative	N/A*	Total		
PVP: Total Formulas Recalled	39/47 (83%)	1/6 (17%)	4/24 (17%)	44/77 (57%)		
Pair-wise Comparisons ^b :						
Positive vs. Negative	83% > 17%***					
Positive vs. N/A	83% > 17%***					
Negative vs. N/A	17% ≠ 17% ** *					
* Includes no response to sentiment or those responses that were either unclear or neutral in regards to sentiment						
***, **, * indicate significance at the 1, 5, and 10 percent, respectively						
^a Chi-Square test is used						
^b Marascuillo procedure is used						

Table 2 also shows that those students who were positively affected by the derivation presentation were better able to recall formulas than those students who were negatively affected.

Sixty-four percent of the positively affected students were able to recall the PVA formula (in various forms) compared to only 17% of those who were negatively affected. The pattern holds for the FVA (PVP) formula with 55% (83%) of the positively affected students able to recall the FVA (PVP) formula compared to only 17% (17%) for those negatively affected. Those who liked the presentation and were able to recall the PVA formula preferred Form C (Table 2.B2), but no significant difference exists between the forms for the FVA formula (Table 2.C2).

CONCLUSION AND IMPLICATIONS

This paper attempts to measure student response to teaching the derivation of annuity formulas in the classroom by observing formula usage as well as surveying student sentiment. Overall, presenting annuity formula derivations seems to benefit students. Student usage of all three forms indicates that some benefit is associated with presenting all three forms although, depending on the type of problem presented, some forms are used more than others and the usage of some forms results in better performance. Also, those students who were positively affected by the derivation presentation are better able to correctly recall formulas than those negatively affected. This result may be interpreted in a variety of ways. On the one hand, since

those who were not able to correctly recall the formulas in any form stated they were negatively affected, an assumption may be drawn that only poorer performing students dislike the derivation presentation. In other words, only the "bad" students do not appreciate "good" teaching. On the other hand, one would expect that those who were negatively affected would not be able to recall the formulas.

Other factors that may affect an instructor's decision to include the derivation lecture include the amount of class time devoted to deriving annuity formulas as well as the benefits and costs of using the various forms throughout the semester. After presenting the formula derivations, an instructor may illustrate how to calculate annuity problems with either all three forms or with only one form. For instance, an instructor may show how to calculate the present value of a bond's coupons (which form an annuity) with each of the three forms or with only Form C as is commonly the case. Of course, using all three forms is time consuming, and may also confuse a few students. With the increasing technology of online teaching platforms, an alternative approach would be to present the derivation of the formulas in video or PowerPoint format as supplemental material via the internet.

In summary, deriving the FVA and PVA annuity formulas results in three common forms of each of the formulas. Results from a quiz requiring the use of one of the three forms for each formula reveals that some forms are used more than others. The form that provides better performance depends on the type of problem presented. In addition, the majority of students indicated a positive sentiment toward presenting the derivation of annuity formulas in the classroom while a small minority disliked the derivation presentation. The results presented in this paper allow the instructor some insight into student perceptions of how annuity formulas should be taught.

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