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Biharmonic timelike curves according to Bishop frame in Minkowski 4-space.

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Abstract

In this paper, we study biharmonic timelike curves according to Bishop frame in Minkowski 4-space E_1^4. Additionally, we give some characterizations for curvatures of these curves with respect to the principal curvature functions $k_1(s)$, $k_2(s)$, $k_3(s)$ according to Bishop frame. Mathematical subject classification (2010): 58B20, 53C40, 53C50.

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Introduction

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results has become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations, because biharmonic maps are solutions of a fourth order strongly elliptic semi linear PDE.

Biharmonic curves $\gamma : I \subset \mathbb{R} \to (N,h)$ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\gamma'}\gamma' - R(\gamma', \nabla^3_{\gamma'}\gamma')\gamma' = 0, \tag{1}$$

where, ∇ is the Levi-Civita connection on (N,h) and R is its curvature operator. As we shall detail in the next section, they arise from a variational problem and are a natural generalization of geodesics. In the last decade biharmonic curves have been extensively studied and classified in several spaces by analytical inspection of Equation 1 [1-15].

Although much work has been done, the full understanding of biharmonic curves in a surface of the Euclidean three-dimensional space is far from been achieved. As yet, we have a clear picture of biharmonic curves in a surface only in the case that the surface is invariant by the action of a one parameter group of isometries of the ambient space. For example, in the study by Caddeo et al. [4] it was proved that a biharmonic curve on a surface of revolution in the Euclidean space must be a parallel that is an orbit of the action of the group on the surface. This property was then generalized to invariant surfaces in a 3-dimensional manifold [15].

In this paper, we study biharmonic non-lightlike curves according to Bishop frame in Minkowski 4-space E_1^4. We give some characterizations for curvatures of a biharmonic non-lightlike curve in E_1^4. This paper is organized as follows: Section 2 gives some basic concepts of the Frenet frame and Bishop frame of a curve in E_1^4. Section 3 obtained some characterizations for curvature of these curves with respect to the principal curvature functions $k_1(s)$, $k_2(s)$, $k_3(s)$ according to Bishop frame.

Preliminaries

The Minkowski space-time is four-dimensional Euclidean space provided with the Lorentzian inner product,

$$u, v_L = -u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

where $u = (u_1, u_2, u_3, u_3)$; $v = (v_1, v_2, v_3, v_3)$ are the rectangular coordinate system of E_1^4. Any vector u in E_1^4 can be characterized as follows: the vector u is called spacelike, lightlike or timelike if $u, u_L > 0$, $u, u_L = 0$ or $u, u_L < 0$ respectively. The norm of a vector $u \in E_1^4$ is given by $u = \sqrt{|u, u_L|}$.

Any arbitrary curve $\gamma = \gamma(s) : I \to E_1^4$ is spacelike, timelike or lightlike (null), if all of its velocity vector $\gamma'(s)$ are spacelike, timelike or lightlike (null), respectively. If $\gamma'(s) = 1$, then γ is called unit speed curve [16]. For any unit speed timelike curve γ with Frenet-Serret frame $\{T, N, B_1, B_2\}$, the Frenet formulas of the curve γ can be given as:

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa(s) & 0 & 0 \\ -\varepsilon_1\kappa(s) & 0 & \varepsilon_3\tau(s) & 0 \\ 0 & -\varepsilon_2\tau(s) & 0 & \varepsilon_4\sigma(s) \\ 0 & 0 & -\varepsilon_3\sigma(s) & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $\varepsilon_1 = T, T_L$, $\varepsilon_2 = N, N_L$, $\varepsilon_3 = B_1, B_{1L}$ and $\varepsilon_4 = B_2, B_{2L}$, and $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are curvature, first and second torsion functions of the curve γ respectively. Here, we call the value of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ as the character of Frenet frame of the curve γ.

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when the curve has vanished second derivative. We can transport an orthonormal frame along a curve simply by Bishoping each component of the frame [1-3,12,13]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) & k_3(s) \\ k_1(s) & 0 & 0 & 0 \\ k_2(s) & 0 & 0 & 0 \\ k_3(s) & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

where $k_1(s)$, $k_2(s)$, $k_3(s)$ are principal curvatures of the timelike curve according to the parallel frame and their expression as following:

$$\begin{cases} k_1(s)=\kappa(s)\cos\alpha sin\beta \,, \\ k_2(s)=\kappa(s)(sin\beta cos\theta-\sin\alpha\cos\beta\, sin\theta), \\ k_3(s)=\kappa(s)(\sin\beta\sin\theta+\sin\alpha\cos\beta\cos\theta), \end{cases}$$

and

$$\kappa(s)=\sqrt{k_1^2(s)+k_2^2(s)+k_3^2(s)}\,,\ \tau(s)=\beta'(s)+\alpha'(s)\tan\alpha\cot\beta$$

$$\sigma(s)=-\frac{\alpha'(s)}{\sin\beta},$$

where β,α,θ are the spherical angle between the spacelike vectors N_3 and N_3 , spacelike vectors N_3 and N_3, spacelike vectors N_3 and N_3 respectively [9]. We shall call the set $\{T,N_1,N_2,N_3\}$ as Bishop frame of γ in the Minkowski 4-space γ .

Discussion and Conclusion

Biharmonic non-lightlike curves according to Bishop frame in E_1^4

In this section we give the notation of biharmonic non-lightlike curves in E_1^4 according to Bishop frame. We also obtain some characterizations for curvature of these curves. Let $\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s),\gamma_4(s))$ be a unit speed timelike curve parametrized by arc-length in E_1^4. The corresponding Bishop frame formula has the following form:

$$\begin{cases} Y'(s)=T, \\ T'(s)=k_1N_1+k_2N_2+k_3N_3, \\ N_1'=k_1T, \\ N_2'=k_2T, \\ N_3'=k_3T, \end{cases} \tag{2}$$

where k_2, k_2 and γ are the principal curvature functions according to Bishop frame of the curve γ . Now, the biharmonic equation for the curve γ reduces to

$$\nabla_T^3T-R(T,\nabla_T^3T)T=0. \tag{3}$$

Here, the curve γ is called a biharmonic curve if it is a solution of the Equation (3).

Theorem 3.1: Let γ be a unit speed timelike curve with non-zero principal curvature k_1, k_2 and k_3 in E_1^4. Then γ is a biharmonic curve if and only if the following conditions are satisfied:

$$\begin{aligned} k_1^2+k_2^2+k_3^2&=\mathcal{V}, \\ k_1''+k_1(k_1^2+k_2^2+k_3^2)&=0, \\ k_2''+k_2(k_1^2+k_2^2+k_3^2)&=0, \\ k_3''+k_3(k_1^2+k_2^2+k_3^2)&=0, \end{aligned} \tag{4}$$

where $\mathcal{V}$ is non-zero constant of integration.

Proof. Using the Bishop Equation (2) and biharmonic Equation (3), we obtain

$$\begin{aligned} &(3k_1k_1'+3k_2k_2'+3k_3k_3')T+\left(k''_1+k_1(k_1^2+k_2^2+k_3^2)\right)N_1 \\ &+\left(k''_2+k_2(k_1^2+k_2^2+k_3^2)\right)N_2+\left(k''_3+k_3(k_1^2+k_2^2+k_3^2)\right)N_3 \\ &-R(T\nabla^3{}_TT)T=0 \end{aligned} \tag{5}$$

In Minkowski 4-space E_1^4, the Riemannian curvature is zero, then we have

$$\begin{aligned} &(3k_1k_1'+3k_2k_2'+3k_3k_3')T+\left(k_1''+k_1(k_1^2+k_2^2+k_3^2)\right)N_1 \\ &+\left(k_2''+k_2(k_1^2+k_2^2+k_3^2)\right)N_2+\left(k_3''+k_3(k_1^2+k_2^2+k_3^2)\right)N_3=0, \end{aligned} \tag{6}$$

By Equation (6), we see that the curve γ is a unit speed biharmonic curve if and only if

$$\begin{aligned} &3k_1k_1'+3k_2k_2'+3k_3k_3'=0, \\ &k_1''+k_1(k_1^2+k_2^2+k_3^2)=0, \\ &k_2''+k_2(k_1^2+k_2^2+k_3^2)=0, \\ &k_3''+k_3(k_1^2+k_2^2+k_3^2)=0, \end{aligned} \tag{7}$$

Hence from the Equation (7), we get the result which complete our proof. □

As a consequence of the Equation (4) we obtain the following Corollary;

Corollary 3.2: Let γ be a unit speed timelike curve with non-zero principal curvature k_1, k_2 and k_3 in E_1^4. Then γ is a biharmonic curve if and only if the following conditions are satisfied:

$$\begin{aligned} k_1^2+k_2^2+k_3^2&=\mathcal{V}, \\ k_1''+k_1\mathcal{V}&=0, \\ k_2''+k_2\mathcal{V}&=0, \\ k_3''+k_3\mathcal{V}&=0. \end{aligned} \tag{8}$$

Where $\mathcal{V}$ is non-zero constant of integration.

Theorem 3.3: Let γ be a unit speed timelike curve with non-zero principal curvature k_1, k_2 and k_3 in E_1^4. Then γ is a biharmonic curve if and only if the following conditions are satisfied:

$$\begin{aligned} &k_1^2(s)+k_2^2(s)+k_3^2(s)=\mathcal{V}, \\ &k_1(s)=A_1\cos\sqrt{\mathcal{V}}s+B_1\sin\sqrt{\mathcal{V}}s, \\ &k_2(s)=A_2\cos\sqrt{\mathcal{V}}s+B_2\sin\sqrt{\mathcal{V}}s, \\ &k_3(s)=A_3\cos\sqrt{\mathcal{V}}s+B_3\sin\sqrt{\mathcal{V}}s, \end{aligned} \tag{9}$$

where A_i,B_j for $1\le i,j\le 3$ and $\mathcal{V}$ are constants of integration.

Proof. Directly, by using Equation (8), we have Equation (9). □

Corollary 3.4: If $A_1=A_2=A_3$ and $B_1=B_2=B_3$, then we have $k_1(s)=k_2(s)=k_3(s)$.

Corollary 3.5: Let γ be a unit speed timelike biharmonic curve with non-zero principal curvature k_1, k_2 and k_3 in E_1^4, then

$$\begin{aligned} &A_1\cos\sqrt{\mathcal{V}}s+B_1\sin\sqrt{\mathcal{V}}s=\sqrt{\mathcal{V}}\sin\Theta\cos\Phi, \\ &A_1\cos\sqrt{\mathcal{V}}s+B_1\sin\sqrt{\mathcal{V}}s=\sqrt{\mathcal{V}}\sin\Theta\sin\Phi, \\ &A_3\cos\sqrt{\mathcal{V}}s+B_3\sin\sqrt{\mathcal{V}}s=\sqrt{\mathcal{V}}\cos\Theta, \end{aligned}$$

where $\Theta \in [0,\pi]$, $\Phi \in [0,2\pi]$ and $\mathcal{V}$, A_i, B_j for $1 \le i, j \le 3$ are constants of integration.

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