

# A numerical solution of inverse scattering problem with non-over-determined data.

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## Abstract

The 3D inverse scattering problem with non-over-determined scattering data is considered. The data are the scattering amplitude  $A(\beta, \alpha_0, k)$  for all  $\beta \in S^2_\beta$ , where  $S^2_\beta$  is an open subset of the unit sphere  $S^2$  in  $\mathbb{R}^3$ ,  $\alpha_0 \in S^2$  is fixed, and  $k \in (a, b), 0 < a < b$ . In this paper, a numerical method is given for solving the inverse scattering problem with non-over-determined scattering data and the numerical results are presented.

**Keywords:** Scattering theory, Obstacle scattering, Uniqueness theorem, Non-over-determined scattering data.

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## Introduction

Let us first consider the direct scattering problem:

$$(\nabla^2 + k^2 - q(x))u = 0 \quad \text{in } \mathbb{R}^3, \tag{1}$$

$$u = e^{ik\alpha \cdot x} + v, \tag{2}$$

$$v = A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \frac{x}{r} = \beta, \tag{3}$$

where  $\alpha, \beta \in S^2$  are the directions of the incident wave and scattered wave correspondingly,  $S^2$  is the unit sphere  $k^2 > 0$  is energy,  $k > 0$  is a constant,  $A(\beta, \alpha, k)$  is the scattering amplitude (scattering data), which can be measured, and  $q(x) \in Q$ , where  $Q$  is a set of  $C^1$ -smooth real-valued compactly supported functions,  $q = 0$  for  $\max_j |x_j| \geq R, R > 0$  is a constant. By  $D$  the support of  $q$  is denoted.

The direct scattering problem (1) - (3) has a unique solution [1].

### Consider now the inverse scattering problem

Find the potential  $q(x) \in Q$  from the scattering data  $A(\beta, \alpha, k)$ . The uniqueness of the inverse scattering problem with fixed-energy data ( $k = k_0 > 0$  is fixed) is proved by Ramm [1]:  $q(x) \in Q$  is uniquely determined by the scattering data  $A(\beta, \alpha, k_0)$  for a fixed  $k = k_0 > 0$  and all  $\alpha, \beta \in S^2$ . Ramm also gave a method for solving the inverse scattering problem with fixed-energy data and obtained an error estimate for the solution for exact data and also for noisy data [1], Chapter 5.

In this paper, we give a numerical implementation of the method proposed by A. G. Ramm, for solving the inverse scattering problem with non-over-determined data, that is, finding  $q(x) \in Q$  from the scattering data  $A(\beta, k) := A(\beta, \alpha_0, k)$  for a fixed  $\alpha_0 \in S^2$ , all  $\beta \in S^2$ , and all  $k \in (a, b), 0 < a < b$ . The basic uniqueness theorem for this problem is proved, see also [2-7]. In Section 2, the idea of this numerical method is described. This idea and the method described in Section 2 belong to A. G. Ramm. In Section 3, the DSM algorithm used to solve ill-posed linear system is described. In Section 4, the numerical procedure is presented and in Section 5, some examples of the numerical inversion are given.

The essential novel features of this inversion method are:

- The data are non-over-determined. So these are minimal scattering data that allow one to uniquely recover the potential.
- The inverse scattering problem is highly non-linear. Nevertheless, our method reduces the inversion to stable solution of linear algebraic system (8).
- The numerical difficulty comes from the fact that system (8) is highly ill-conditioned.

### Inversion method

Let  $q \in Q$ . The unique solution to (1) - (3) solves the integral equation:

$$u(x, k) = e^{ik\alpha_0 \cdot x} - \int_D g(x, y, k) q(y) u(y, k) dy, \quad g(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{4}$$

$k > 0$  is a constant,  $\alpha_0 \in S^2$  is fixed. This equation is uniquely solvable for  $u$  under our assumptions on  $q$ . Let  $h(x, k) = q(x)u(x, k)$ . Then (4) implies

$$h(x, k) = q(x)e^{ik\alpha_0 \cdot x} - q(x) \int_D g(x, y, k) h(y, k) dy. \tag{5}$$

Equations (3) and (4) yield the exact formula for the scattering amplitude:

$$-4\pi A(\beta, k) = \int_D e^{-ik\beta \cdot y} h(y, k) dy. \tag{6}$$

From equation (5) one derives the formula for  $q(x)$  if  $h(x, k)$  is found:

$$q(x) = h(x, k) [e^{ik\alpha_0 \cdot x} - \int_D g(x, y, k) h(y, k) dy]^{-1}. \tag{7}$$

### The idea of our inversion method

By the uniqueness theorem from [6], equation (6) has a unique solution  $h(x, k)$ . Solve numerically equation (6) for  $h(x, k)$ . To do this, discretize (6) and get a linear algebraic system for  $h_{pm} := h(y_p, k_m)$ . If  $h_{pm}$  are found, then  $q(x_p)$  are found by formula (7), see also formula (9) below.

Let us partition  $D$  into  $P$  small cubes with volume  $\Delta_p, 1 \leq p \leq P$ . Let  $y_p$  be any point inside the small cube  $\Delta_p$ . Choose  $P$  different numbers  $k_m \in (a, b), 1 \leq m \leq P$ , and  $P$  different vectors  $\beta_j \in S^2, 1 \leq j \leq P$ . Then discretize (6) and get a linear algebraic system for finding  $h_{pm}$ :

$$-4\pi A(\beta_j, k_m) = \sum_{p=1}^P e^{-ik_m \beta_j \cdot y_p} h_{pm} \Delta_p, \quad 1 \leq j, m \leq P, \tag{8}$$

Where  $h_{pm} := h(y_p, k_m)$ . Solve the linear algebraic system (8) numerically, then use equation (7) to find the values of the unknown potential  $q(x_p)$ :

$$q(x_p) = h_{pm} \left[ e^{ik_m \alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^P g(x_p, y_{p'}, k_m) h_{p'm} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P. \tag{9}$$

**Remark 1:** Note that the right hand side of (9) should not depend on  $m$  or  $j$ . This independence is an important requirement in the numerical solution of our inverse scattering problem, a compatibility condition for the data. This requirement is automatically satisfied for the limiting integral equation (6), see formula (7).

The values of  $q(y_p)$  essentially determine the unknown potential  $q(x)$  if  $P$  is large and  $q$  is continuous. The potential  $q$  is unique by the uniqueness theorem in [6].

Note that one can choose  $\beta_j$  and  $k_m$  so that the determinant of the system (8) is not equal to zero [7], so that the system (8) is uniquely solvable, but the numerical difficulty is unavoidable: the system (8) is very ill-conditioned because it comes from an integral equation (6) of the first kind with an analytic kernel. We use the dynamical system method (DSM) from [2] to solve stably the ill-conditioned system (8).

**Dynamical System Method (DSM)**

Equation (8) with noisy data is a linear algebraic system of the form

$$Mu_\delta = f_\delta \tag{10}$$

where  $M$  is the matrix of the size  $P \times P$  and  $f_\delta$  is the noisy data, that is, the noisy values of  $-4\pi A(\beta_j, k_m)$  in equation (8),  $\|f_\delta - f\| = \delta$  for a fixed  $m$  and  $1 \leq j \leq P$ . To solve this ill-posed system we use the Dynamical Systems Method (DSM) from [2]:

$$u_\delta(t) = -u_\delta(t) + (T + a(t))^{-1} M^* f_\delta, \quad u(0) = u_0, \tag{11}$$

where  $T := M^* M$ ,  $M^*$  is the adjoint matrix and  $a(t) > 0$  is a non-increasing function such that  $a(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Equation (11) has a unique solution

$$u_\delta(t) = u_0 e^{-t} + e^{-t} \int_0^t e^s (T + a(s))^{-1} M^* f_\delta ds. \tag{12}$$

To use the DSM method, we need to choose  $a(t)$  and find a stopping time  $t_\delta$  so that  $u_\delta(t_\delta)$  approximates the solution of  $Au = f$ , so that  $\lim_{\delta \rightarrow 0} \|u_\delta - u\| = 0$ , where the norm is in  $\mathbb{R}^P$ .

**Choice of  $a(t)$  and  $t_\delta$ :**

In [2], necessary conditions for  $a(t)$  are:  $a(t)$  is a nonincreasing function and  $\lim_{t \rightarrow \infty} a(t) = 0$ . In our experiments, we choose

$$a(t) = \frac{1}{1+t}. \text{ Consider a step function } \tilde{a}(t) \text{ approximating } a(t): \tilde{a}(t) = a_n, \quad t_n \leq t < t_{n+1}, \tag{13}$$

where  $a_n = a(t_n)$  and we choose  $t_n$  such that  $t_{n+1} - t_n = h_n$ ,  $h_n = q^n$ ,  $q = 2$ . From equation (12), one computes  $u_n = u(t_n)$  by:

$$u_n = u_0 e^{-t_n} + e^{-t_n} \sum_{i=1}^n (e^{t_i} - e^{t_{i-1}}) (T + a_{i-1})^{-1} M^* f_\delta. \tag{14}$$

and one gets an iterative method to solve (12):

$$u_{n+1} = e^{-h_n} u_n + (1 - e^{-h_n}) (T + a_n)^{-1} M^* f_\delta, \quad h_n = t_{n+1} - t_n. \tag{15}$$

To choose  $t_\delta$ , we use a relaxed discrepancy principle [2]:

$$0.9\delta \leq \|Mu_n - f_\delta\| \leq 1.001\delta, \tag{16}$$

where the norm is the vector norm:  $\|\tilde{x}\| := \sqrt{x_1^2 + \dots + x_p^2}, \tilde{x} := (x_1, \dots, x_p) \in \mathbb{R}^p$ . One stops the iterative method (15) when condition (16) is satisfied.

**Numerical Procedure and Results**

In practice, one can measure the noisy scattering data experimentally. For our numerical experiments, we need to construct the noisy scattering data  $A(\beta_j, k_m)$ .

**Constructing noisy scattering data**

Given a potential  $q(x)$ , let us first construct the exact scattering data  $\tilde{A}(\beta_j, k_m)$ . We partition  $D$  into  $P$  small cubes and discretize equation (4) to get

$$u(x_p, k_m) = e^{ik_m \alpha_0 \cdot x_p} - \sum_{j=1}^P g(x_p, y_j, k_m) q(y_j) u(y_j, k_m) \Delta_j. \tag{17}$$

One solves this linear system for  $u(x_p, k_m)$  assuming that  $q(x)$  is known. Then the exact scattering data are calculated by the following formula:

$$\tilde{A}(\beta_j, k_m) = -\frac{1}{4\pi} \sum_{p=1}^P e^{-ik_m \beta_j \cdot y_p} q(y_p) u(y_p, k_m) \Delta_p. \tag{18}$$

Then one can randomly perturb each  $\tilde{A}(\beta_j, k_m)$ ,  $1 \leq j \leq P$  by  $\tilde{\delta} = \text{const} > 0$  to get the noisy scattering data  $A(\beta_j, k_m) = \tilde{A}(\beta_j, k_m) \pm \tilde{\delta}$  where the plus or minus sign is randomly chosen. Then  $\delta := \|A(\beta_j, k) - \tilde{A}(\beta_j, k)\| := \sqrt{\sum_{j=1}^P |A(\beta_j, k) - \tilde{A}(\beta_j, k)|^2}$ , for a fixed  $k$ , is the noise level of the data in equation (10).

**Numerical procedure**

The following steps are implemented in each experiment:

- Choose  $D, \alpha_{op}, P, q(x), k_m$ , and  $\tilde{\delta}$ .
- Use (18) to find the exact scattering data  $\tilde{A}(\beta_j, k_m)$ ,  $1 \leq j \leq P$ . Try different values of  $k_m$  so that the determinant of the system in (8) is not zero. Let  $k$  be the found value of  $k_m$ .
- Use the procedure in Section 4.1 to get the noisy scattering data  $A(\beta_j, k)$ ,  $1 \leq j \leq P$ .
- Solve the linear algebraic system (8) to get  $h_{pm} = h(y_p, k_m)$ . Here we use the DSM method in Section 3 with the noise:

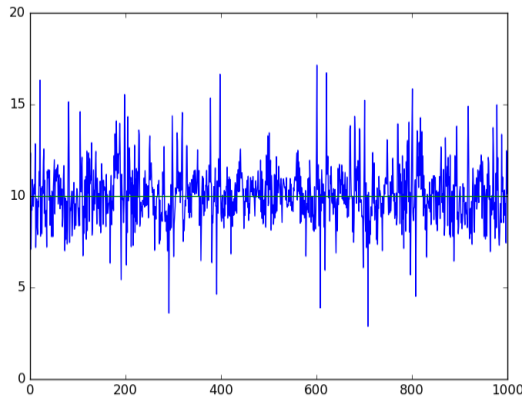
$$\delta := \|A(\beta_j, k_m) - \tilde{A}(\beta_j, k_m)\| := \sqrt{\sum_{j=1}^P |A(\beta_j, k_m) - \tilde{A}(\beta_j, k_m)|^2}, \quad 1 \leq j \leq P.$$

**Table 1.** Numerical results for constant potential  $q(x) = 10$

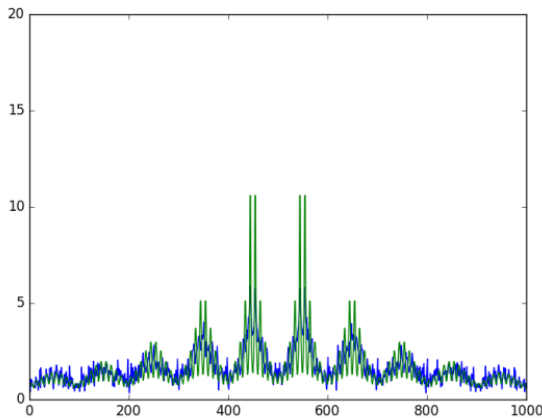
Constant Potential $q(x) = 10$		
$\tilde{\delta}$	$\delta = \ A(\beta_j, k) - \tilde{A}(\beta_j, k)\ $	Relative Error
.04	0.4348	0.0566
.02	0.2174	0.0037
.01	0.1087	0.00065

**Table 2.** Numerical results for the potential  $q(x) = \frac{\exp(-|x|)}{|x|}$ .

Potential $q(x) = \frac{\exp(- x )}{ x }$		
$\tilde{\delta}$	$\delta = \ A(\beta_j, k) - \tilde{A}(\beta_j, k)\ $	Relative Error
.04	0.0806	0.1284
.02	0.0403	0.0547
.01	0.0201	0.0367



**Figure 1.** Constructed potential vs original constant potential  $q(x) = 10$  when  $\tilde{\delta} = 0.01$ .



**Figure 2.** Constructed potential vs original potential  $q(x) = \frac{\exp(-|x|)}{|x|}$  when  $\tilde{\delta} = 0.01$ .

- Calculate the potential  $\tilde{q}(x_p)$  by formula (9):

$$\tilde{q}(x_p) = h_p \left[ e^{ik\alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^P g(x_p, y_{p'}, k_m) h_{p'm} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P.$$

See Remark 1 below formula (9).

- Find the relative error of the estimate of the potential:

$$\text{err} := \frac{\|q(x) - \tilde{q}(x)\|}{\|q(x)\|} = \sqrt{\frac{\sum_{p=1}^P |q(x_p) - \tilde{q}(x_p)|^2 \Delta_p}{\sum_{p=1}^P |q(x_p)|^2 \Delta_p}}. \quad (19)$$

**Numerical Results**

In these experiments, we choose  $D$  to be the unit cube centered at the origin, with sides parallel to coordinate planes,  $P = 1000$ ,  $\alpha_0 = (1, 0, 0)$ , and  $50 \leq k_m \leq 100$  is chosen so that the determinant of the system in (8) is not zero. The condition number of  $A$  is  $\sim 10^{13}$ .

In the following graphs, the  $x$ -axis is the index of the collocation points (this index varies from 1 to 1000) and the  $y$ -axis is the value of the potential.

**Constant potential with compact support**

The inverse scattering problem with constant in  $D$  potential is used to test our inversion method. In this experiment, we take  $q(x) = 10$ . The results are obtained as in Table 1 and Figure 1.

**Potential  $q(x) = \frac{\exp(-|x|)}{|x|}$**

In this experiment, we take  $q(x) = \frac{\exp(-|x|)}{|x|}$ . The results are obtained as in Table 2 and Figure 2.

**Conclusion**

In this paper, a numerical method is given for solving the inverse scattering problem with non-over-determined scattering data and the numerical results are presented.

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